A Problem Kernelization for Graph Packing

Hannes Moser

Friedrich-Schiller-Universität Jena, Germany
Institut für Informatik

SOFSEM 2009

H-Packing

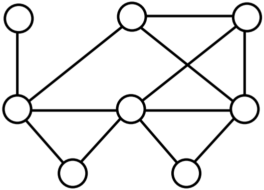
H-Packing Problem

Input: An undirected graph $G = (V, E)$ and a parameter k.

Question: Exist at least k vertex-disjoint copies of H in G?

Example

$H =$ 



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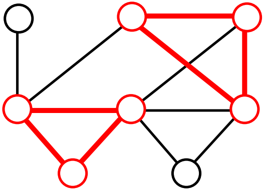

Known Results

- Polynomial-time solvable if H is a graph with at most two vertices
- NP-complete if H is a connected graph with at least three vertices [KIRKPATRICK & HELL, STOC 1978]

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- Randomized algorithm with running time $2^{|V(H)|k} \cdot \text{poly}(n)$ [Koutis, ICALP 2008]
- Deterministic algorithm with running time $12.8^{|V(H)|k} \cdot \text{poly}(n)$ [Kneis, Mölle, Richter, Rossmanith , WG 2006] [Chen, Lu, Sze, Zhang, SODA 2007]

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- H contains three vertices: deterministic algorithm with running time $3.52^{3k} \cdot \text{poly}(n)$ [Wang & Feng, TAMC 2008]
- H is a path on three vertices: deterministic algorithm with running time $2.48^{3k} \cdot \text{poly}(n)$ [Fernau & Raible, COCOA 2008]

Known Results (2)

If H is a triangle:

- Factor-$(3/2 + \epsilon)$ polynomial-time approximation
  [Hurkens & Schrijver, SIAM Journal on Discrete Mathematics, 1989]
- APX-hard in general graphs, factor-1.2 polynomial-time approximation on graphs with maximum degree four
  [Manic & Wakabayashi, Discrete Mathematics, 2008]
- NP-hard to approximate within ratio 139/138
  [Chlebík & Chlebíková, CIAC 2003]

Fixed-Parameter Tractability

Definition

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Problem Kernel

Parameterized problem L. Instance (I, k).

$$(I, k) \xrightarrow[\text{poly}(n) \text{ time}]{\text{data reduction rules}} (I', k')$$

- $(I, k) \in L \leftrightarrow (I', k') \in L$
- $k' \leq k$
- $|I'| \leq g(k)$

Kernelization Results

Known Results

- H is a triangle: $O(k^3)$-vertex kernel
  [Fellows, Heggernes, Rosamond, Sloper, Telle, WG 2004]
- H is a path on three vertices: $7k$-vertex kernel
  [Wang, Ning, Feng, Chen, TAMC 2008]
- H is a star with d leaves: $((d^3 + 4d^2 + 6d + 4) \cdot k)$-vertex kernel [Fellows, Guo, Moser, Niedermeier, STACS 2009]

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New Result

- $O(k^{|V(H)|-1})$-vertex kernel for H-Packing

Kernel for Triangle Packing

Our approach combines ideas for problem kernels for

- SET PACKING and generalized matching problems [FELLOWS ET AL., Algorithmica, 2007], and
- HITTING SET [ABU-KHZAM, WADS 2007].

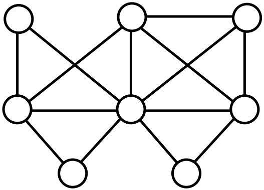
Kernel for Triangle Packing

Data Reduction Rule

Remove all vertices and edges that are not contained in any triangle in G.

Witness Structure $\mathcal{T}$

A *witness structure* is a maximal set $\mathcal{T}$ of triangles in G that pairwise intersect in at most one vertex.



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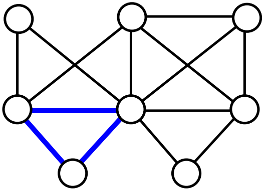

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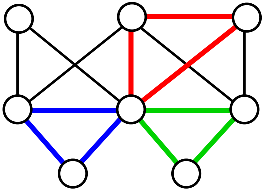

Bounding the Size of $\mathcal{T}$

Data Reduction Rule

If there is a vertex that is contained in at least $3k - 2$ triangles in $\mathcal{T}$, then delete it from G and set $k := k - 1$.

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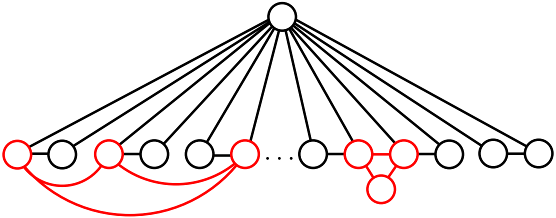

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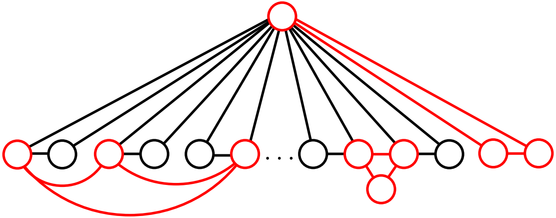


Bounding the Size of $\mathcal{T}$

Lemma

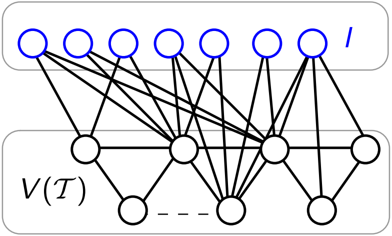
The witness structure $\mathcal{T}$ contains $O(k^2)$ triangles.

Proof

- Suppose that there is no packing of k triangles and let P be a maximum packing of triangles.
- P contains at most $k-1$ triangles.
- Each vertex of each triangle is contained in at most $3k-3$ triangles of $\mathcal{T}$.
- There are at most $3 \cdot (k-1) \cdot (3k-3) = O(k^2)$ triangles in $\mathcal{T}$.

Bounding the Size of the Remaining Graph

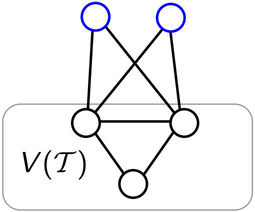
Structure



- $I := V \setminus V(\mathcal{T})$ forms an independent set.
- Each triangle that contains a vertex in I shares two vertices with a triangle in $\mathcal{T}$.

Bounding the Size of the Remaining Graph

General Idea



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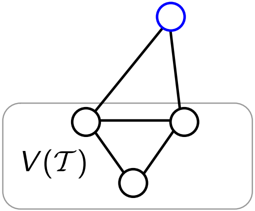

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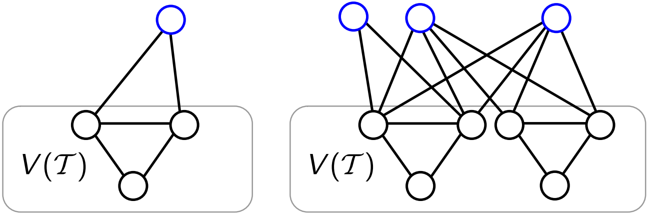

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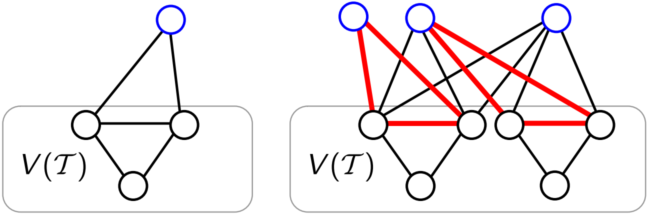

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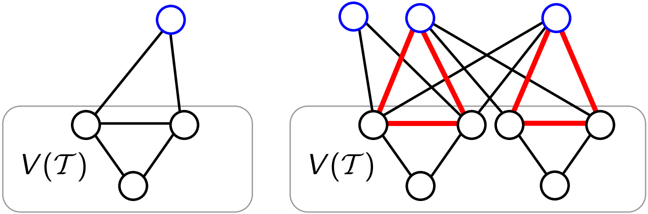

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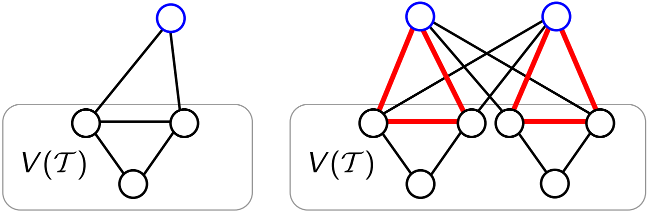
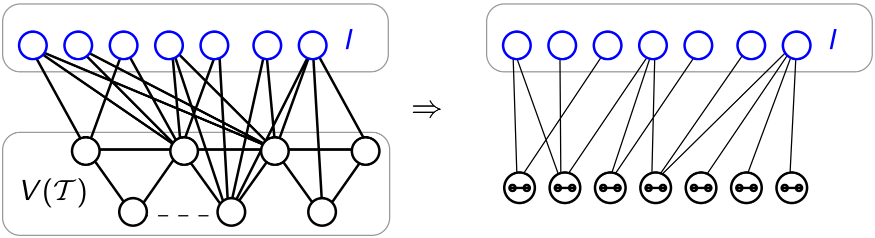


Bounding the Size of the Remaining Graph

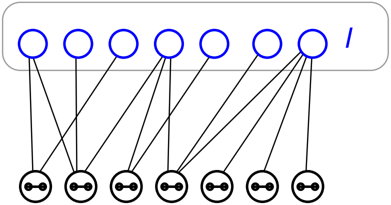
Auxiliary Graph



Bounding the Size of the Remaining Graph

Data Reduction Rule

Compute a maximum matching in the auxiliary graph and remove all unmatched vertices in I.



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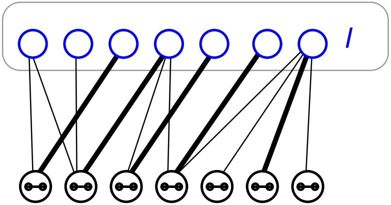

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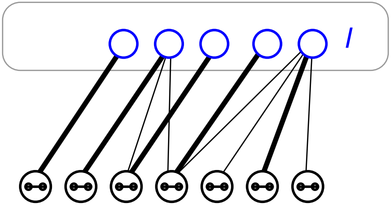

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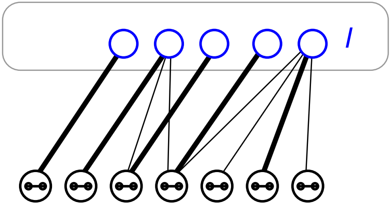

After removing the unmatched vertices, there are no more vertices in I than edges in triangles in $\mathcal{T}$.

$\Rightarrow$ There are $O(k^2)$ vertices in I.

Summary

- The witness structure contains $O(k^2)$ triangles.
- There are $O(k^2)$ vertices contained in these triangles.
- The number of remaining vertices is bounded by $O(k^2)$.
- All the steps can be performed in polynomial time.
- Thus, we obtain a $O(k^2)$-vertex kernel.

Further Result and Open Questions

Further Result

- The approach can be generalized to H-Packing.

Open Questions

- Does Triangle Packing admit a *size*-$O(k^2)$ kernel?
- "Ultimate Goal": Does Triangle Packing admit an $O(k)$-vertex kernel?

Thank you!