Parameterized Complexity of Finding Regular Induced Subgraphs

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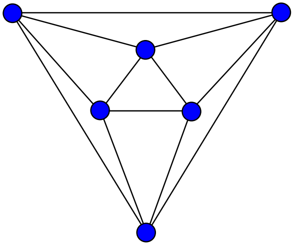
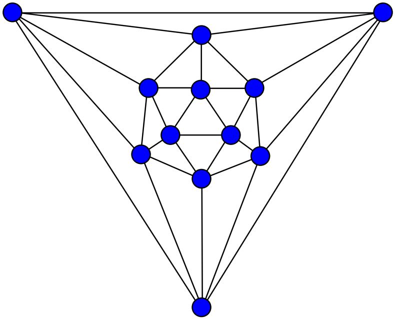
ACiD 2006

Regular Graphs

Definition (Regular Graph)

regular:	All vertices have the same degree.
r-regular:	Every vertex has degree r.





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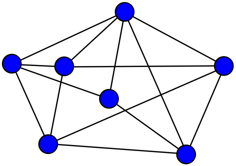
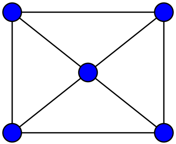
r-REGULAR SUBGRAPH: Can we make a graph r-regular by vertex/edge deletions?

r-REGULAR INDUCED SUBGRAPH: Can we make a graph r-regular by vertex deletions?





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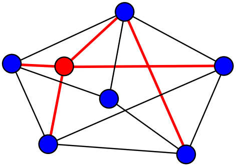

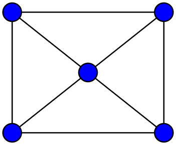

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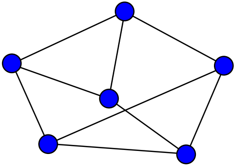

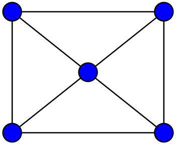

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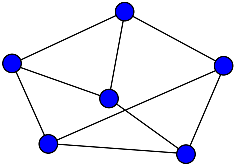

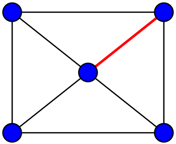

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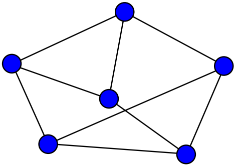

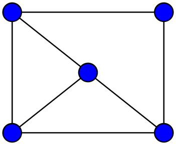

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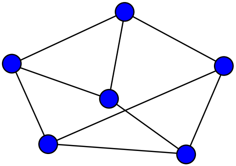

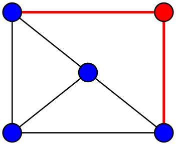

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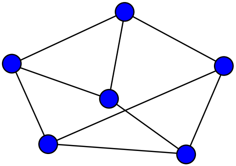

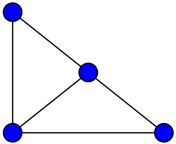

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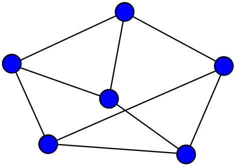

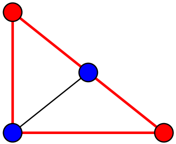

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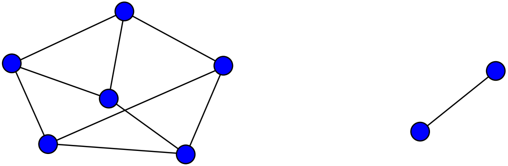

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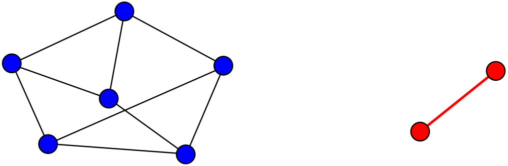

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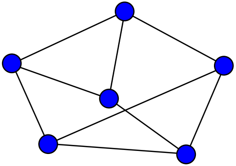

Known Results

NP-completeness of subgraph problems

- CUBIC SUBGRAPH

  [GAREY AND JOHNSON, 1979]
- CUBIC SUBGRAPH on planar graphs with maximum degree 7

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- r-REGULAR SUBGRAPH on planar graphs ($r \geq 3$)

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We are considering **induced** subgraphs.

Vertex Deletion Problems

Can we delete at most k vertices such that the resulting graph has a certain property? (cycle-free, chordal, 2-colorable, **regular**, ...)

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Known Result

Vertex Deletion is NP-complete for hereditary properties.

[Lewis and Yannakakis, Journal of Computer an System Sciences 20, 1980]

The Parameterized Complexity Approach to NP-Hard Problems

We want an exact algorithm, but this implies exponential running time.

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Fixed-Parameter Intractability

The basic complexity class for fixed-parameter intractability is $W[1]$. Parameterized Reductions are used to show $W[1]$-hardness.

Parameterized Complexity of Vertex Deletion Problems

Vertex Deletion with Hereditary Properties

- Hereditary property can be characterized by forbidden induced subgraphs: FPT

  [Cai, Information Processing Letters 58, 1996]

- Hereditary property includes all trivial graphs but not all complete graphs or vice versa: W[1]-hard, FPT otherwise

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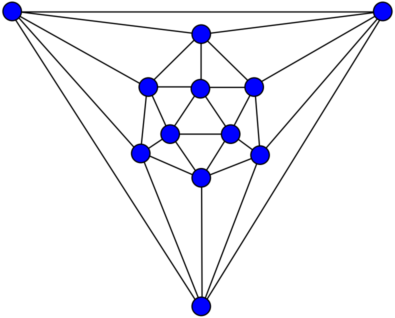
Regularity is not a hereditary property!

Example



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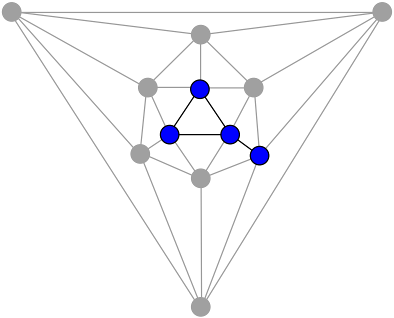


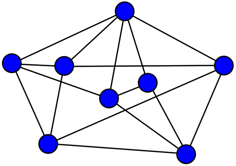
k-Almost r-Regular Graph

Input

An undirected graph $G = (V, E)$ and a nonnegative integer k.

Question

Is there a vertex subset $S \subseteq V$ of size at most k such that $G[V \setminus S]$ is r-regular?



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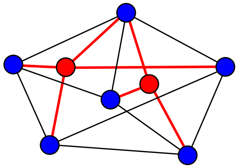

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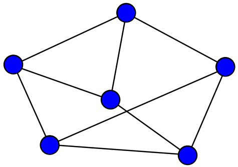

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Remarks

- For $r = 0$, the problem is equivalent to Vertex Cover.
- For $r = 0$, the dual parameterization is equivalent to Independent Set.
- For $r = 1$, the dual parameterization is equivalent to Induced Matching.

Main Results

1. k-Almost r-Regular Graph is NP-complete on triangle free planar graphs.

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Remaining Talk

Show 2.

Kernel

Approach to show fixed-parameter tractability

Provide data reduction rules that lead to a problem kernel (in polynomial time).

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Problem Kernel

Parameterized problem L. Instance (I, k).

$$(I, k) \xrightarrow[O(n^{O(1)})]{\text{reduction rules}} (I', k')$$

- $(I, k) \in L \leftrightarrow (I', k') \in L$
- $k' \leq k$
- $|I'| \leq g(k)$

A Problem Kernel for k-Almost r-Regular Graph

Theorem

The k-Almost r-Regular Graph problem, for $r \geq 1$, has a kernel of size $O(kr \cdot (k + r)^2)$.

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Main Idea

Replace big r-regular connected subgraphs with smaller ones.

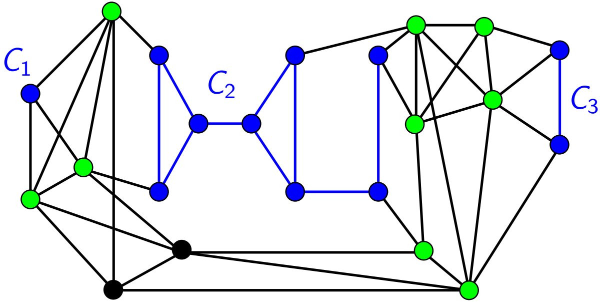
Definitions

Definition (Clean Region)

A vertex is clean if it has degree r. A clean region is a maximal subset of clean vertices that induces a connected subgraph in G.

Definition (Boundary)

The boundary B_i of a clean region C_i is its open neighborhood.



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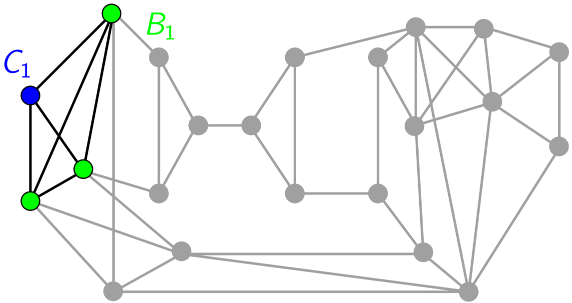

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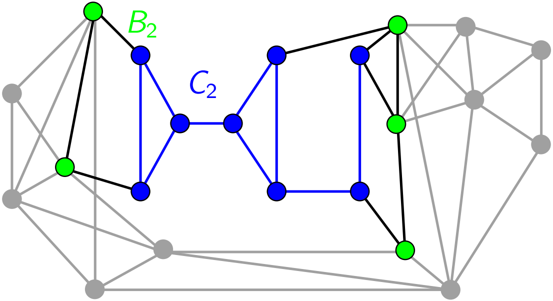

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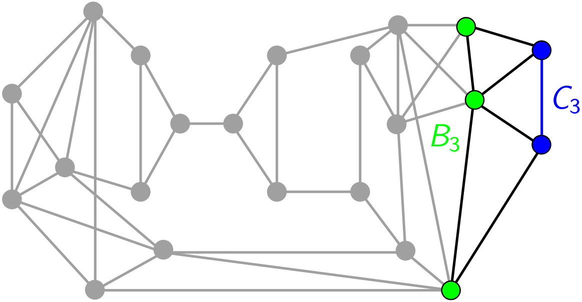
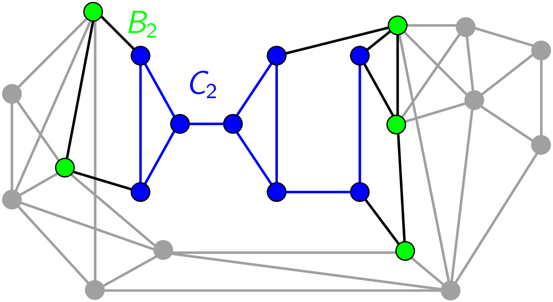
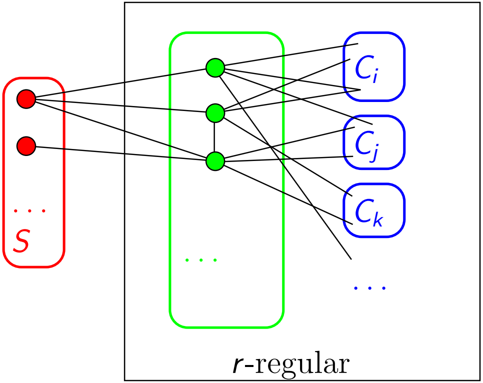


Important Observation

If $v \in S$ for some $v \in B_i \cup C_i$, then $C_i \subseteq S$.



Graph Structure



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Apply a series of reduction steps such that the resulting graph satisfies the following properties:

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Apply a series of reduction steps such that the resulting graph satisfies the following properties:

1. All vertices have degree at least r and at most $k + r$,
2. each vertex of a boundary B_i has at most r neighbors in C_i,
3. for every clean region C_i, $|C_i| \leq \max\{k + 1, (r + 1) \cdot |B_i|\}$.

Property 2

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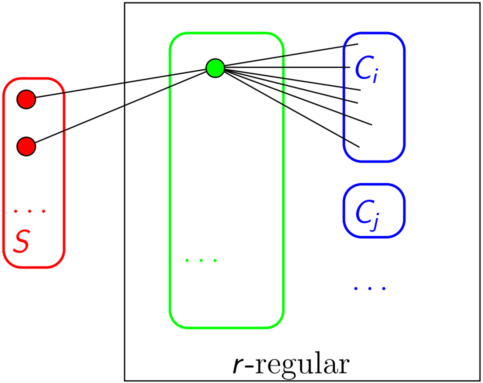
Intuitive Idea

All vertices in the open neighborhood of a solution S are not clean. If such a vertex had too many neighbors in a clean region (not in S), then S would not be a solution.



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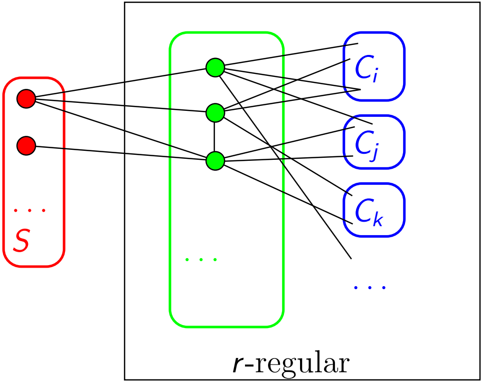
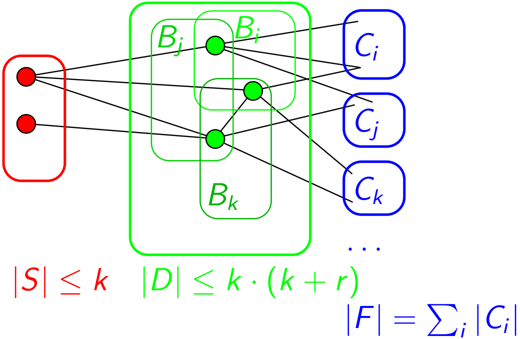
Intuitive Idea

Big clean regions cannot be a part of the solution. We can replace them by smaller (but not too small) clean regions.



Kernel Size



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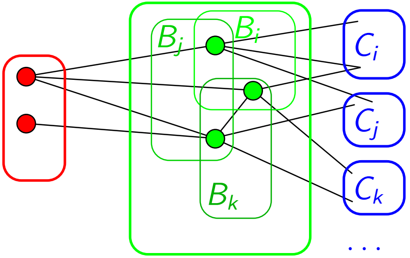

$|S| \leq k$ $\quad |D| \leq k \cdot (k + r)$

$|F| = \sum_i |C_i|$

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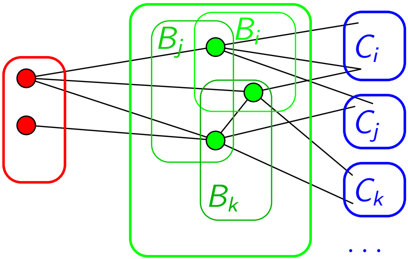

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$\Rightarrow |S| + |D| + |F| \leq O(kr \cdot (k + r)^2)$

Future Work and Open Questions

- r part of the input problem?
- Parameterized complexity of other non-hereditary properties?
- Can there be derived more general results for non-hereditary properties?

Thank you!