A Generalization of Nemhauser and Trotter's Local Optimization Theorem

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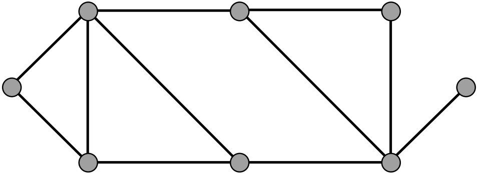
The Vertex Cover Problem

Vertex Cover

Input: An undirected graph $G = (V, E)$ and a parameter $k \geq 0$.

Question: Can we find a vertex set $S \subseteq V$, $|S| \leq k$, such that each edge has a least one endpoint in S.

Example



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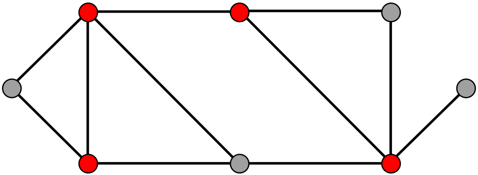
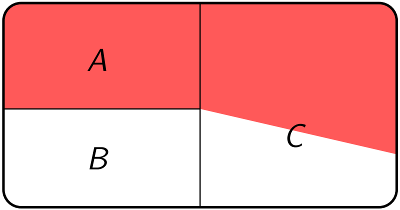


Nemhauser and Trotter's Local Optimization Theorem

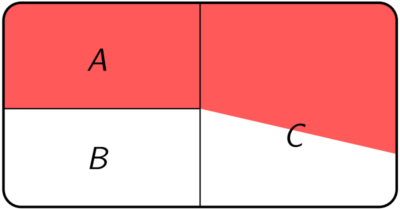
NT-Theorem [Nemhauser & Trotter, Math. Program. 1975]

For $G = (V, E)$ one can compute in polynomial time a partition of V into three subsets A, B, and C:



1. There is a min.-cardinality vertex cover S of G with $A \subseteq S$
2. If S' is a vertex cover of $G[C]$, then $A \cup S'$ is a vertex cover of G
3. Every vertex cover of $G[C]$ has size at least $|C|/2$

Consequences



- $A \cup C$ is a factor-2 approximate vertex cover of G.
- $G[C]$ is a $2k$-vertex problem kernel for Vertex Cover.

Fixed-Parameter Tractability and Problem Kernel

Fixed-Parameter Tractability

A parameterized problem with input instance (I, k) is *fixed-parameter tractable* with respect to parameter k if it can be solved in $f(k) \cdot \text{poly}(|I|)$ time.

Problem Kernel

$$(I, k) \xrightarrow[\text{poly}(|I|)\text{ time}]{\text{data reduction rules}} (I', k')$$

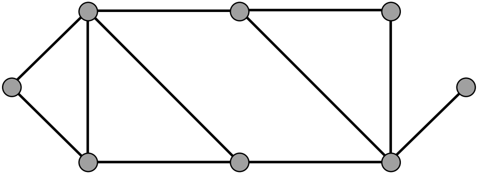
- $(I, k) \in L$ if and only if $(I', k') \in L$,
- $k' \leq k$, and
- $|I'| \leq g(k)$ for some function g

d-Bounded-Degree Deletion

Input: An undirected graph $G = (V, E)$ and a parameter $k \geq 0$.

Question: Can we find a vertex set $S \subseteq V$, $|S| \leq k$, such that each vertex in $G[V \setminus S]$ has degree at most d?

Example for $d = 2$



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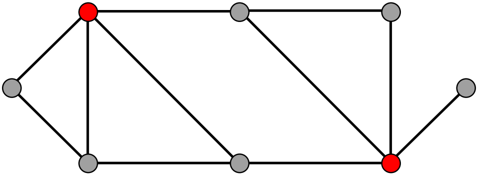

Generalizing Vertex Cover

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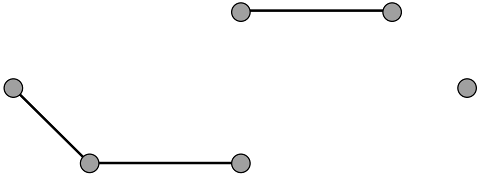

Motivation: Finding Dense Subgraphs

- Finding max.-cardinality cliques is an important task in Bioinformatics
- Successful approach: Transform to the dual Vertex Cover problem

  [Chesler et al., Nature Genetics, 2005]

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- Drawback: cliques are overly restrictive

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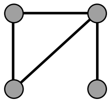
- Drawback: cliques are overly restrictive
- Use s-plexes instead of cliques

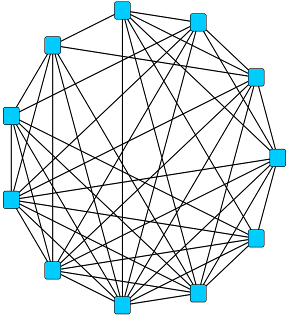
s-plex

A graph is an s-plex if each vertex is adjacent to all but $\leq s - 1$ vertices.

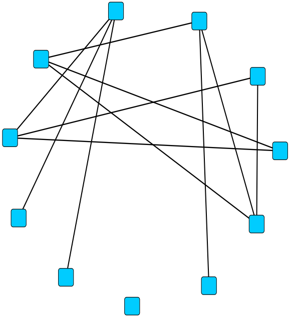


3-plex

Motivation: Finding Dense Subgraphs



Maximum-cardinality 4-plex in fission yeast protein-protein interaction network



Corresponding complement

(Data source: www.thebiogrid.org)

Known Results for d-Bounded-Degree Deletion

- NP-complete for all $d \geq 0$

  [Lewis and Yannakakis, J. Comput. System Sci., 1980]

- Can be solved in time $O((d+k)^{k+1} \cdot n)$

  [Nishimura, Ragde, Thilikos, Discrete Appl. Math., 2005]

- Enumeration of all minimal solutions in time $O((d+2)^k \cdot (k+d)^2 \cdot m)$

  [Komusiewicz, Hüffner, Moser, Niedermeier, Theor. Comput. Sci.]

- Problem kernel of size $15k$ for $d = 1$
  Problem kernel of size $O(k^2)$ for constant $d \geq 2$

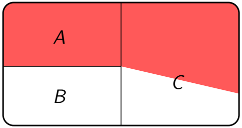
  [Prieto and Sloper, Theor. Comput. Sci., 2006]

- Experimental study for $d = 0$ (Vertex Cover)

  [Abu-Khzam et al., ALENEX 2004]

“NT-Theorem” for d-Bounded-Degree Deletion

For $G = (V, E)$ one can compute in polynomial time a partition of V into three subsets A, B, and C:



1. There is a min.-cardinality solution S for G with $A \subseteq S$
2. If S' is a solution for $G[C]$, then $A \cup S'$ is a solution for G
3. Every solution for $G[C]$ has size at least

$$\frac{|C|}{d^3 + 4d^2 + 6d + 4}$$

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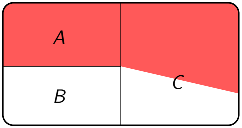

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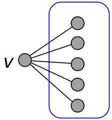
$\Rightarrow$ $G[C]$ is a $(d^3 + 4d^2 + 6d + 4) \cdot k$-vertex problem kernel.

High-Degree Reduction Rule

If there exists a vertex $v \in V$ with $\deg(v) > d + k$, then delete v and set $k := k - 1$.



Low-Degree Reduction Rule

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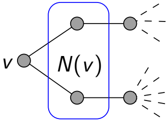

$O(k^2)$-Vertex Problem Kernel for d-Bounded-Degree Del.

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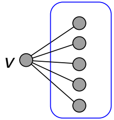

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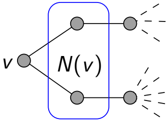
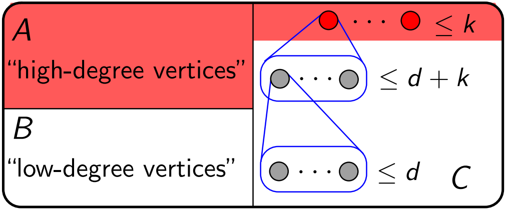


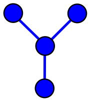


$\Rightarrow O(k^2)$-vertex kernel for constant d

Basic Observation

A star with $d+1$ leaves is a forbidden subgraph for graphs of maximum degree d.

$d = 2$



A Linear-Vertex Kernel

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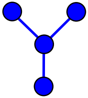
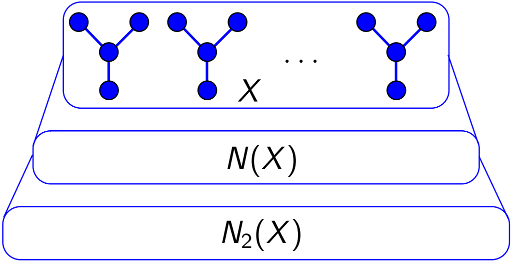


First Step of Kernelization

Find a maximal collection of vertex-disjoint copies of a star with $d + 1$ leaves.



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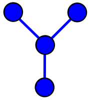
$d = 2$ 

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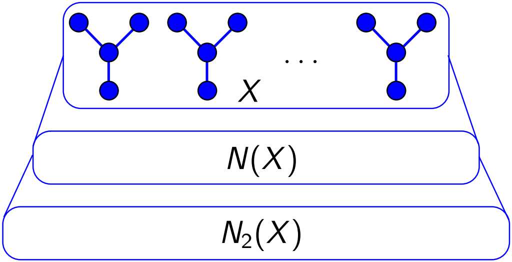
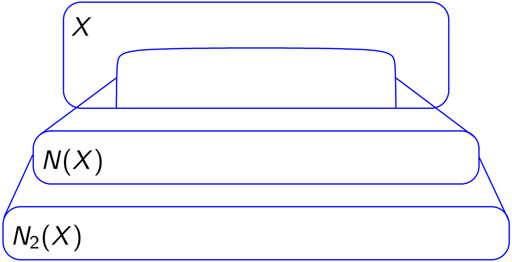


- $G[V \setminus X]$ has maximum degree d
- There are $\leq k$ stars in the collection
- $|X| = O(k)$

Ideal Situation



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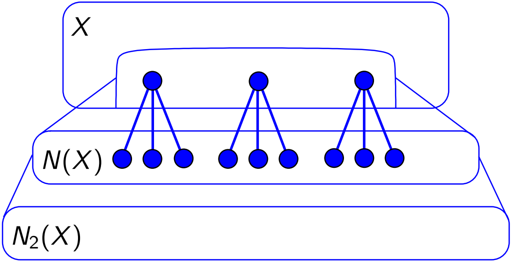

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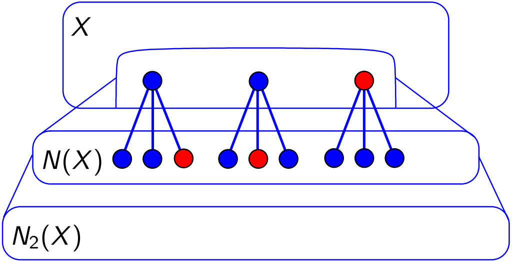

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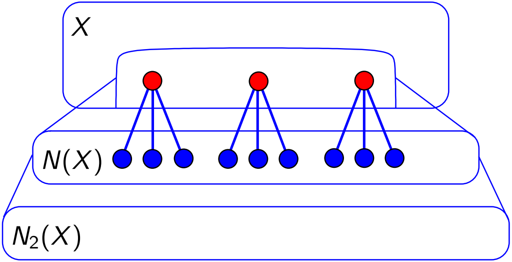

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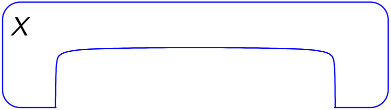
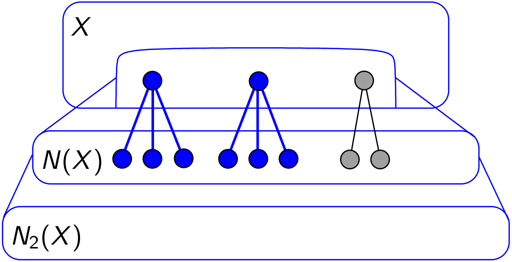


Main Idea for Linear-Vertex Kernel



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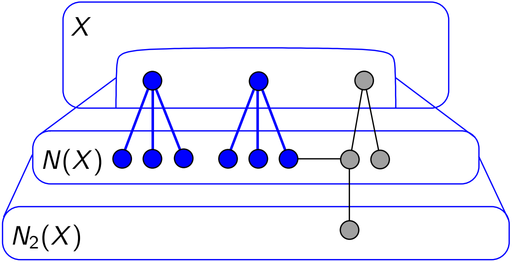

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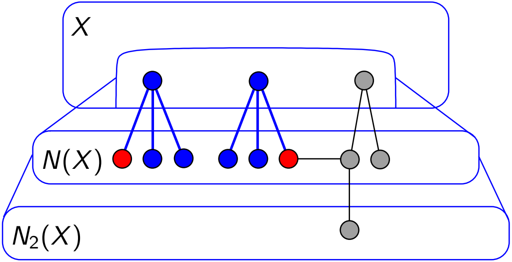

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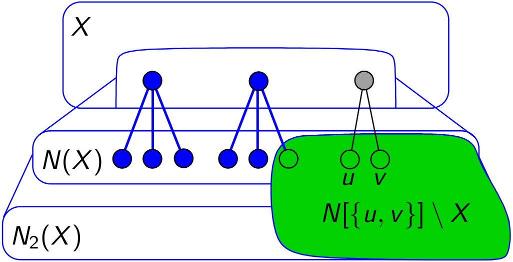

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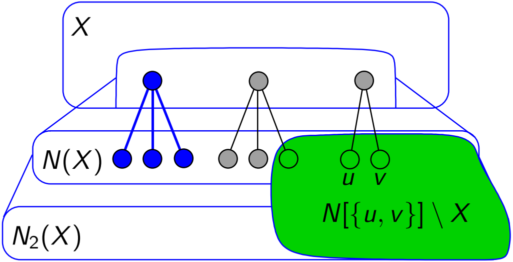

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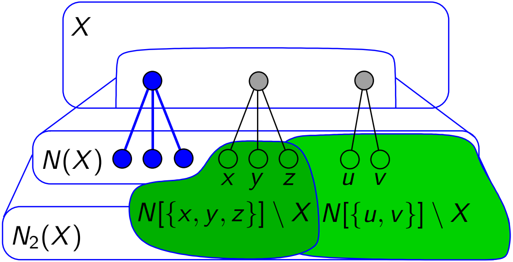

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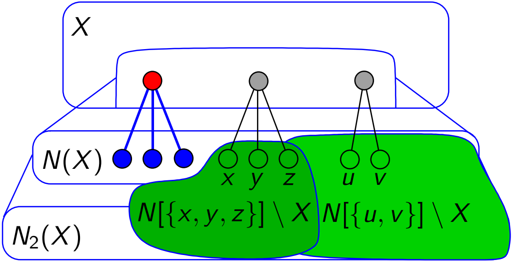

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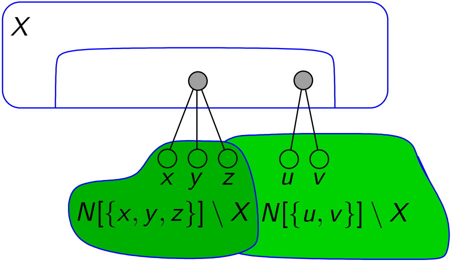

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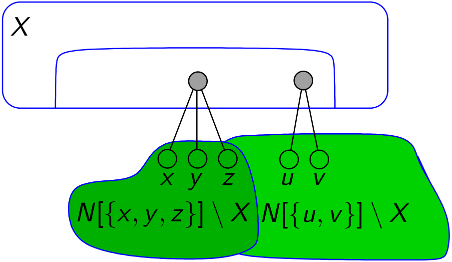

Observation

For each gray vertex in X there are at most $d \cdot (d+1)$ green vertices in $V \setminus X$.

$\Rightarrow$ The remaining graph contains $O(k)$ vertices for constant d.
$\Rightarrow$ d-Bounded-Degree Deletion admits an $O(k)$-vertex problem kernel.

Outlook

Further Results

- Bounded-Degree Deletion is $W[2]$-complete for unbounded d.
- Implementation and experiments
  [Moser, Niedermeier, Sorge, Manuscript, submitted]

Future Research

- Further improvement of the kernel size.
- For which other problems does this technique work?

Thank you!