Iterative Compression for Solving Hard Network Problems

Hannes Moser

Institut für Informatik
Friedrich-Schiller-Universität Jena

Project ITKO (NI 369/5)
DFG Schwerpunktprogramm 1126 – Jahreskolloquium 2006

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Example

Vertex Cover can be solved in time $O(1.2852^k + k|V|)$.
k: size of the vertex cover

Techniques to Show Fixed-Parameter Tractability

Established Techniques

- Kernelizations
- Depth-bounded search trees
- Dynamic Programming
- Tree Decompositions
- ...

Recent Approach

- Iterative compression

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Inductive approach: Compute a solution for a problem instance using the information provided by a solution for a smaller instance.

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Example for this talk

FEEDBACK VERTEX SET IN TOURNAMENTS.

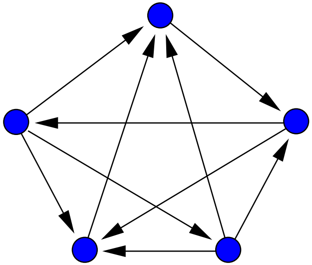
[DOM, GUO, HÜFFNER, NIEDERMEIER, AND TRUSS, CIAC 2006]

Feedback Vertex Set in Tournaments

Definition (Feedback Vertex Set in Tournaments)

Input: Tournament $G = (V, E)$, integer $k \geq 0$.

Output: Is there a subset $X \subseteq V$ of at most k vertices such that $G[V \setminus X]$ has no cycles?



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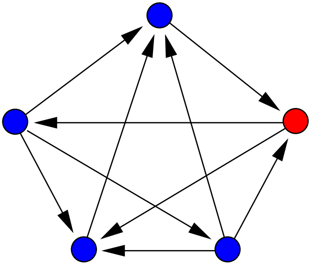

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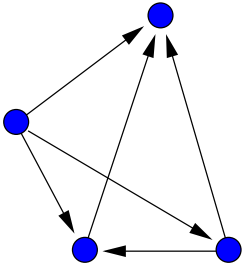

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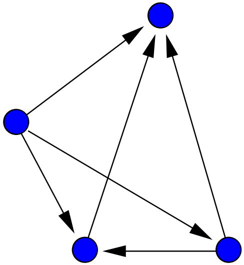

Feedback Vertex Set in Tournaments is NP-complete

[Speckenmeyer, WG 1989]

Iterative Compression Framework

Idea

Use a *compression routine* iteratively: Given a solution of size $k+1$, compute a solution of size k.

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Iterative Compression Framework for Feedback Vertex Set in Tournaments

1. Start with an empty graph G' and an empty solution X.
2. For each vertex v in G:
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   2.3 If $|X| > k$, then return "NO".

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Invariant during the loop:
X is a solution of size at most k for the current graph G'.

Compression Routine (1)

Task

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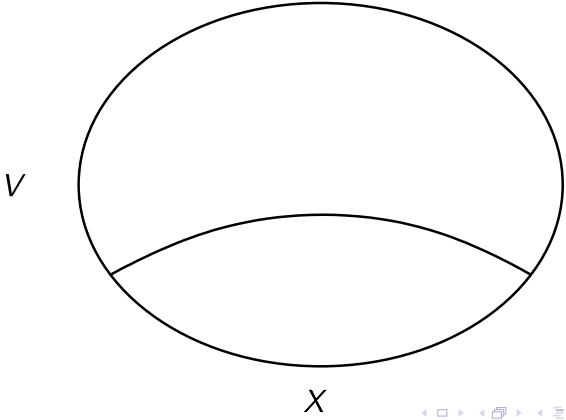
Try all $2^{|X|}$ partitions of X into a part to exchange (S) and a part to keep ($X \setminus S$).



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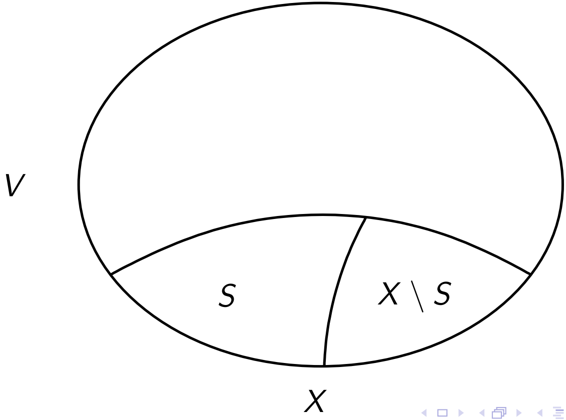

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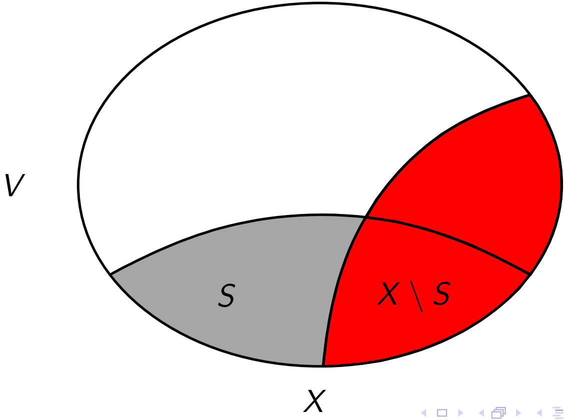

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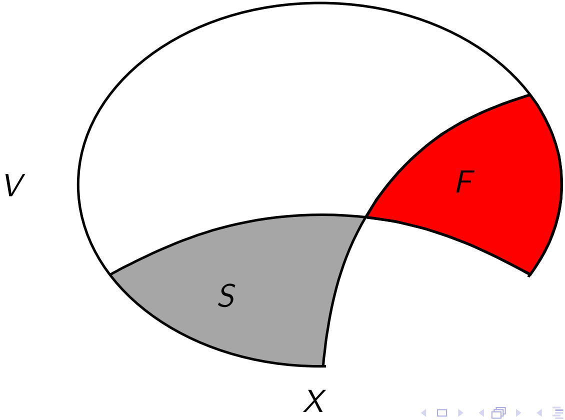

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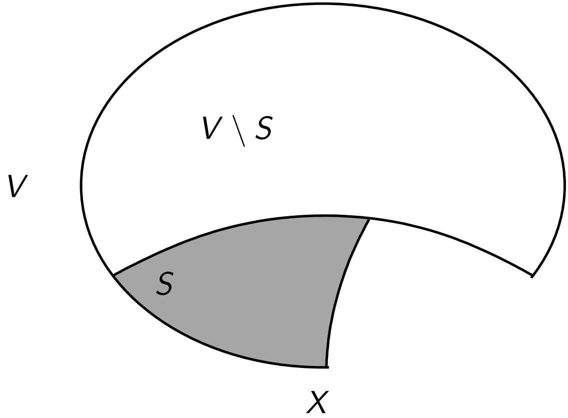

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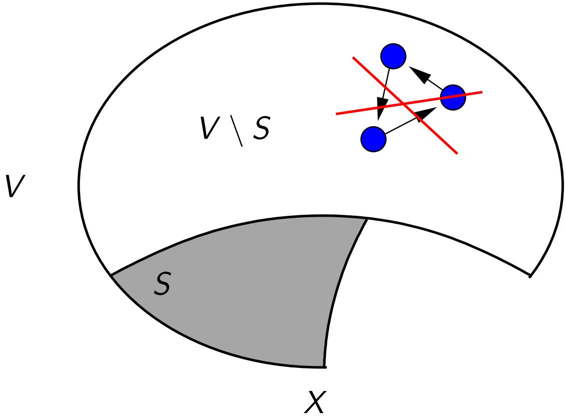

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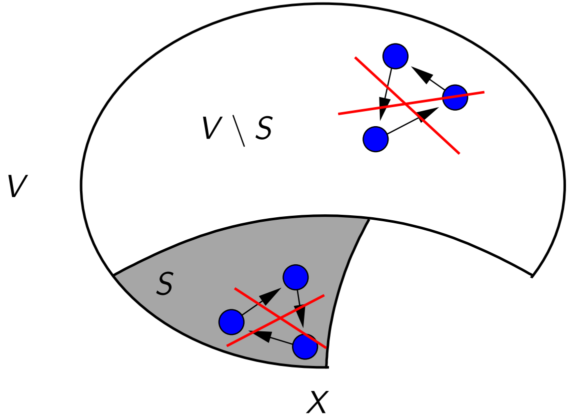

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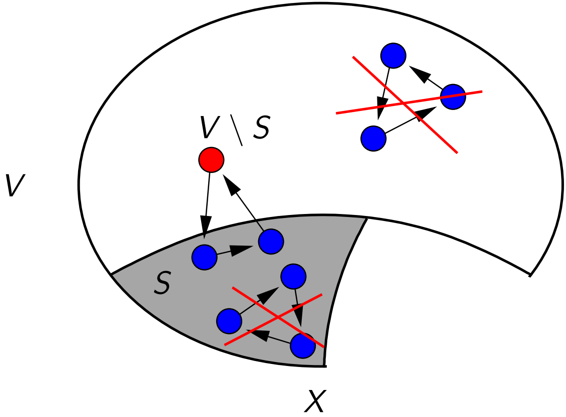
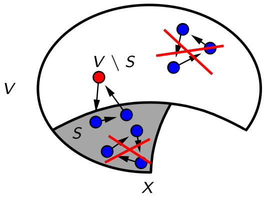


(Lemma: A tournament contains a cycle iff it contains a triangle.)

Topological Sorts

- $G[S]$ is cycle-free $\Rightarrow$
  S has topological sort $s_1, \ldots, s_{|S|}$.
- $G[V \setminus S]$ is cycle-free $\Rightarrow$
  $V \setminus S$ has topological sort.



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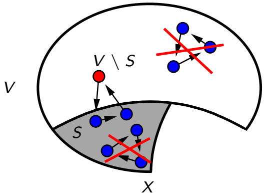

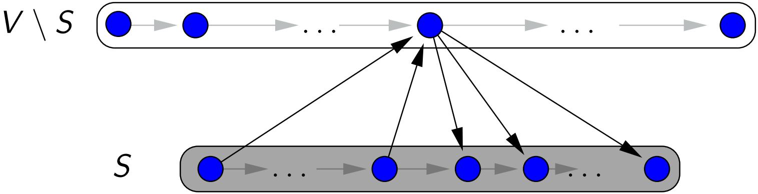

Goal: Insert a maximum subset of $V \setminus S$ into $s_1, \ldots, s_{|S|}$.

Observation: Every vertex v of $V \setminus S$ has a "natural position" $p(v)$ relative to the vertices of S.

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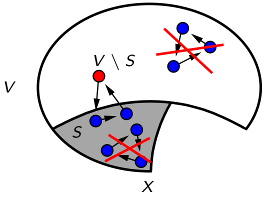

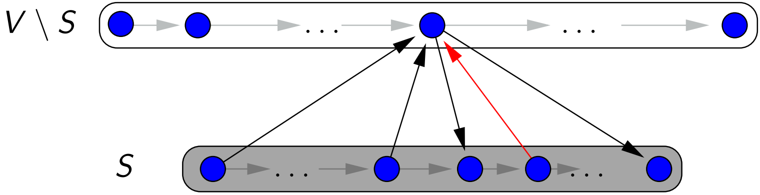

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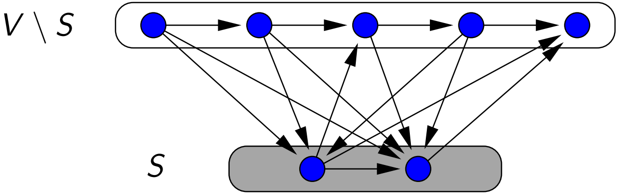

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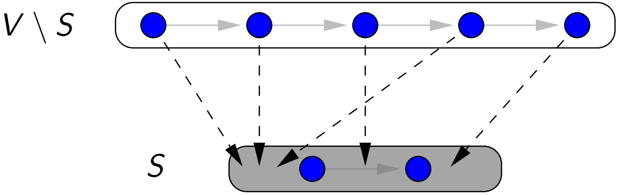

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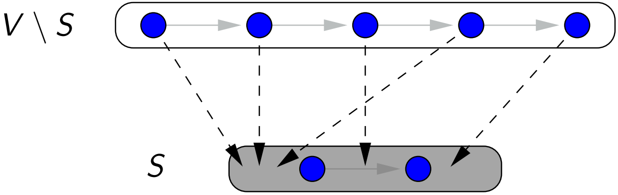


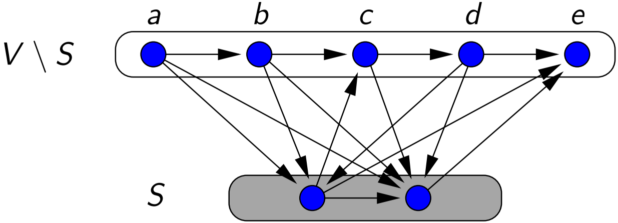
The vertices $(V \setminus S) \setminus F$ must be sortable in such a way that

- the topological sort of $V \setminus S$ is preserved, and
- the sort of $V \setminus S$ by "natural position" p is preserved.

$\Rightarrow$ Search for the longest common subsequence of both sorts.

Example

- $V \setminus S$ sorted topologically: *abcde*
- $V \setminus S$ sorted by p: *abdce*
- A longest common subsequence is *abce*



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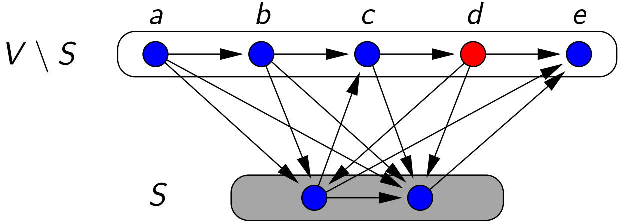

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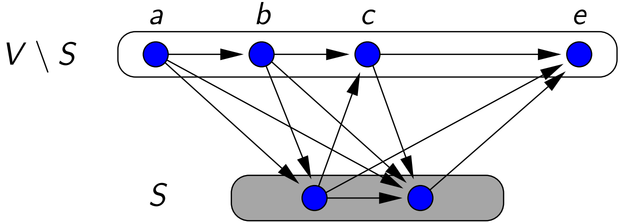

Overall Running Time

Time consumption:

- n iterations of the compression routine;
- 2^{k+1} partitions per iteration;
- time $O(n \cdot k)$ for destroying triangles, time $O(n \log n)$ for sorting vertices and finding the longest common subsequence.

Running time for solving FEEDBACK VERTEX SET IN TOURNAMENTS:

$$O(2^k \cdot n^2(\log n + k))$$

Applications of Iterative Compression

- Graph Bipartization $O(3^k kmn)$
  [Reed, Smith, and Vetta, Operations Research Letters 32, 2004]
- Edge Bipartization $O(2^k m^2)$
  [Guo, Gramm, Hüffner, Niedermeier, and Wernicke, WADS 2005]
- Feedback Vertex Set $O(c^k m)$
  [Dehne, Fellows, Langston, Rosamond, and Stevens, COCOON 2005]
  [Guo, Gramm, Hüffner, Niedermeier, and Wernicke, WADS 2005]
- Feedback Vertex Set in Tournaments $O(2^k n^2(\lg n + k))$
  [Dom, Guo, Hüffner, Niedermeier, and Truss, CIAC 2006]
- Chordal Deletion
  [Marx, WG 2006]
- Implementation of Graph Bipartization $O(3^k mn)$
  [Hüffner, WEA 2005]

Discussion

Advantages

- The problem becomes easier: We improve a solution instead of computing an optimal solution directly.
- Optimize results from approximation algorithms and heuristics.

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Future Work

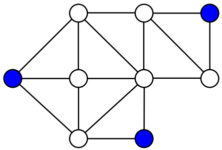
- Characterize problems amenable to solution compression.
- How can iterative compression be combined with other techniques?
- Apply to maximization problems.

A Candidate Problem for Future Research

Minimum Vertex Multiway Cut

Input: A graph $G = (V, E)$, a set of terminals $T \subseteq V$, and an integer $k \geq 0$.

Output: Is there a subset $X \subseteq V$ of at most k vertices such that no two vertices of T belong to the same connected component of $G[V \setminus X]$?



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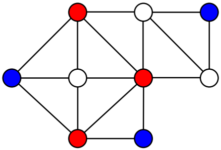

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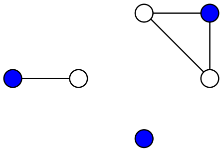

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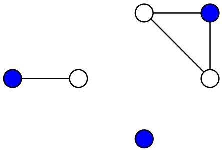

Known Results

- NP-complete [CUNNINGHAM, DIMACS Series in DM and TCS, vol. 5, 1991]
- Fixed-parameter tractable $O(c^{k^3} \cdot \text{poly}(n))$ [MARX, IWPEC 2004]

Thank you!